Example 1. An urn contains 100 marbles: 60 blue marbles and 40 red marbles. A marble is drawn from the urn, what is the **probability** that the marble is blue?

Assumption: Each marble is just as likely to be drawn as any other.

Question: Why is this a fair assumption?

Answer: The probability of an event reflects what we know (and don't know) about the event. Given our information, the simplest assumption is that all the marbles are equally likely to be drawn... *Because we have no reason to conclude otherwise.*

This principle is called the *principle of insufficient reason* (or the *principle of indifference*).

With more information or more experience with this urn of marbles, we may eventually reconsider this assumption.

Conclusion: There are 60 blue marbles and 100 marbles total, all of which are equally likely to be drawn, the probability of drawing a blue marble is equal to the proportion of blue marbles in the urn, which is $60/100 = 60\%$.

Shorthand: $P(E) =$ the probability of the event E.

Viewing probability as a *proportion*, we note that...

(*) If an event E is certain to occur, then $P(E) = 100\%$.

(*) If E is certain to ***not*** occur, then $P(E) = 0\%$.

(*) In all other cases, $0\% < P(E) < 100\%$.

$\Rightarrow$ The closer $P(E)$ is to 100% the more certain we are that E will occur.

$\Rightarrow$ The closer $P(E)$ is to 0% the more certain we are that E will not occur.

Observation: the event "*E does **not** occur*" is called the *complement* of E. Whenever we perform an experiment or procedure that might lead to the occurrence of E, then ***exactly one*** of E or *not* E must occur. I.e., their probabilities must add up to 100%:

$$P(E) + P(\mathit{not}\ E) = 100\% \implies \boxed{P(\mathit{not}\ E) = 100\% - P(E)}$$

Example 2. Suppose that the marbles in Example 1 are marked with the letter A or B. Specifically,

- 40 blue marbles are marked with A; 20 blue marbles are marked with B.
- 10 red marbles are marked with A; 30 red marbles are marked with B.

A marble is drawn *at random* from the urn. What is the probability that it is marked with an A?

First: What does '*at random*' imply here? *At random* means that on each draw, each of the tickets in the box has an equal chance of being drawn.

$\Rightarrow$ There are 50 marbles marked with an A, so $P(A) = 50/100 = 50\%$.

A marble is drawn from the urn, and it is observed to be blue. What is the probability that it is marked with an A?

We now have more information: the marble is known to be blue. We can ignore the red marbles and imagine that the marble was drawn from an urn of 60 blue marbles, 40 of which are marked with an A.

$\Rightarrow$ $P(A \;\; \textit{given that the marble is blue}) = 40/60 \approx 66.67\%$.

Definition. The probability of event E given that we know that event F has occurred is called the ***conditional probability of E given F***.

(*) Shorthand $P(E|F)$ = conditional probability of E given F.

E.g., in Example 2, we found that $P(A|\text{ blue marble}) \approx 66.67\%$.

(*) The probability of E (*without* any additional information) is called the *unconditional* probability of E.

Example 3. A marble is drawn from the urn in Example 2 and you are told that it is marked with an A. What is the probability that the marble is blue?

$\Rightarrow$ There are 50 marbles marked with an A and 40 of these are blue, so

$$P(\text{blue marble}|A) = 40/50 = 80\%.$$

Example 4. What is the probability that a marble drawn from the (same) urn is red, if we know that it is marked with a B?

$\Rightarrow$ There are 50 marbles marked with a B of which 30 are red, so

$$P(\text{red marble}|B) = 30/50 = 60\%.$$

Example 5. What is the probability that a marble drawn at random from the (same) urn is red ***and*** marked with a B?

$\Rightarrow$ There are 100 marbles in the urn and 30 of them are red B s, so

$$P(\text{red } \textit{and } B) = 30/100 = 30\%.$$

Or...

$\Rightarrow$ 40% of the marbles in the urn are red, and 75% of the red marbles are marked with a B, so

$$P(\text{red } \textit{and } B) = (40\%) \times (75\%) = 30\%.$$

In other words

$$P(\text{red } \textit{and } B) = P(\text{red}) \times P(B|\text{red}).$$

The Multiplication rule. *Given two events E and F*

$$P(E \text{ and } F) = P(E) \times P(F|E)$$

Also

$$P(E \text{ and } F) = P(F \text{ and } E) = P(F) \times P(E|F)$$

Example 6. Two cards are dealt from the top of a well-shuffled deck. What is the probability that the first card is a King and the second card is a 7?

$\Rightarrow$ The probability that the first card is a King is $4/52$ and the probability that the second card is a 7 given that the first card is a King is $4/51$, so

$$\begin{aligned} &P(\text{first card King } \textit{and} \text{ second card } 7) \\ &\qquad = P(\text{first card King}) \times P(\text{second card } 7 | \text{first card King}) \\ &\qquad = \frac{4}{52} \cdot \frac{4}{51} = \frac{16}{2652} \approx 0.6\% \end{aligned}$$

Example 6a. Three cards are dealt from the top of a well-shuffled deck. What is the probability that the first card is a Jack and the third card is a 4?

$\Rightarrow$ The probability that the first card is a Jack is $4/52$ and the probability that the third card is a 4 given that the first card is a Jack is $4/51$, so

$$\begin{aligned} &P(\text{first card Jack } \textit{and} \text{ third card } 4) \\ &\qquad = P(\text{first card Jack}) \times P(\text{third card } 4 | \text{first card Jack}) \\ &\qquad = \frac{4}{52} \cdot \frac{4}{51} = \frac{16}{2652} \approx 0.6\% \end{aligned}$$

Observation: The second card is unknown and so for all intents and purposes it is just another card in the deck.

Example 7. A card is dealt from the top of a well-shuffled deck, then it is replaced, the deck is reshuffled and another card is dealt. What is the probability that the second card is a 7 given that the first card is a King?

$\Rightarrow$ Since the first card was replaced (and the deck was reshuffled) before the second card was dealt, the identity of the first card doesn't provide any information about the identity of the second card, so

$$P(\text{second card 7}|\text{first card King}) = P(\text{second card 7}) = \frac{4}{52} \approx 7.7\%.$$

Definition. If $P(E|F) = P(E)$, then the events E and F are said to be (statistically) ***independent***.

Comment: *Independence* is not the same as unrelated. Two events can be closely related, but *statistically* independent.

Example 8. A box contains 200 tickets...

- 120 tickets are marked with an X and 80 tickets are marked with a Y.
- Of the X-tickets, 30 are also marked with an A and the other 90 are marked with a U.
- Of the Y-tickets, 20 are also marked with an A and the other 60 are marked with a U.

One ticket is drawn at random from the box...

(*) There are 200 tickets overall and $50 = 30 + 20$ of them are marked with an A, so $P(A) = 25\%$.

(*) There are 120 X-tickets and 30 of them are marked with an A, so $P(A|X) = 30/120 = 25\%$.

(*) The events A and X are *independent* (but not unrelated).

Comments.

(1) If the events E and F are independent, then $P(E|F) = P(E)$ (and $P(F|E) = P(F)$). In this case the multiplication rule reduces to

$$P(E \text{ and } F) = P(E)P(F).$$

In fact, this formula can be used as the definition of independence. I.e., we can say '*E and F are independent if* $P(E \textit{ and } F) = P(E)P(F)$.'

(2) Independence is often an ***assumption*** that we make about the events in question because it makes calculations easier.

(3) ***But*** the assumption of independence must be justified and reevaluated if subsequent results point in another direction. One of the easiest ways to misuse probability is to make an unjustified assumption of independence. See the example in Section 13.5 of the textbook.

Box models.

Many questions in probability can be answered by considering an appropriate *box model*.

A box model is comprised of two components.

(i) A (hypothetical) box of tickets, each of which is labelled in various ways, and

(ii) A number of draws from the box.

We will imagine drawing tickets from the box in one of two ways.

(*) ***With replacement*** – after a ticket is drawn and observed, it is replaced in the box. In this case the composition of the box doesn't change from draw to draw, and the results of the different draws are ***independent***.

(*) ***Without replacement*** – each ticket that is drawn from the box stays out of the box for the remaining draws. In this case, the composition of the box changes from draw to draw, and the results of the different draws are ***dependent***.

Example 9. A box contains 50 tickets: 20 red tickets, 15 blue tickets, 10 green tickets and 5 orange tickets.

(*) If 3 tickets are drawn from the box at random ***with*** replacement, what is the probability that all three of the tickets are blue?

Now: The results of the draws are independent, so

$$\begin{aligned} &P(\text{1st blue and 2nd blue and 3rd blue}) \\ &\qquad = P(\text{1st blue}) \cdot P(\text{2nd blue}) \cdot P(\text{3rd blue}) \\ &\qquad = \frac{15}{50} \cdot \frac{15}{50} \cdot \frac{15}{50} = 2.7\% \end{aligned}$$

Example 9a. If 3 tickets are drawn from the same box at random ***without*** replacement, what is the probability that all three of the tickets are blue?

$\Rightarrow$ results of the draws are ***not*** independent, so

$$
\begin{aligned}
&P(\text{1st blue and 2nd blue and 3rd blue})\\
&\quad = P((\text{1st blue and 2nd blue}) \text{ and 3rd blue})\\
&\quad = P(\text{1st and 2nd blue}) \cdot P(\text{3rd blue}|\text{1st and 2nd blue})\\
&\quad = \overbrace{P(\text{1st blue}) \cdot P(\text{2nd blue}|\text{1st blue})}^{P(\text{1st and 2nd blue})} \cdot P(\text{3rd blue}|\text{1st and 2nd blue})\\
&\quad = \frac{15}{50} \cdot \frac{14}{49} \cdot \frac{13}{48} = 2.32\%
\end{aligned}
$$

Question (to grow on):

Two tickets are drawn from the the same box at random, with replacement.

What is the probability that the two tickets have different colors?